Existence Of Special Lagrangian Sphere On Kummer Surface

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1 Introduction

In Paul Seidel's thesis, he showed that if a compact Kahler manifold admits an ordinary double point, then it contains a lagrangian sphere. While Stenzel[1] contructed a Calabi-yau metric on cotangent bundle of n-dimensional sphere and the zero section of it is special lagrangian with respect to the metric. Those two facts motivate us to ask whether it's true for Calabi-yau manifold with ordinary double point to contain a special lagrangian sphere. We're going to show that it's true for Kummer surface. The reason why it's comparably easy for Kummer surface is that we can deform Eguchi-hanson metric, which is a special case of Stenzel metric, by a small amount to a new complete calabi-yau metric. And the "difference" between those two metrics is controlled. Thus we can apply Mclean's theorem to show that zero section is still special lagrangian with respect to the calabi-yau matric on Kummer surface. We'll start from technical materials which will be used in deforming Eguchi-hanson metric, and then follow Simon Donaldson's gluing argument[2] and then show the existence of special lagrangian sphere.

2 Technical background[2]

The section is aimed for preparation of deforming the Eguchi-hanson metric on Kummer surface.[2]

2.1 Analysis

We want to apply the theory of translation-invariant elliptic operators on cylinders $M \times \mathbf{R}$, where M is a compact Riemannian $(n-1)$-manifold. Consider Δ defined to be the standard Laplace operator of the product metric and we use the sign convention that Δ is a positive operator. For $p > 1$ and integers $k \geq 0$, define Sobolev spaces L^p_k on $M \times \mathbf{R}$ by taking the completion of the smooth compactly supported functions under the usual norm. Then we have

Proposition 1 *For any p, k the map $\Delta + 1 : L^p_k \to L^p_k$ is an isomorphism.*

Take $p = 2$. Assuming standard results about the compact manifold M. Given a smooth function ρ of compact support we want to solve the equation $(\Delta+1)f = \rho$. We can do this by separation of variables, writing $\rho = \sum \rho_\lambda(t)\phi_\lambda$ where ϕ_λ is an orthonormal basis of eigenfunctions of the Laplacian Δ_M on M—so $(\Delta+1)\phi_\lambda = (1+\lambda)\phi_\lambda$. We seek a solution $f = \sum f_\lambda(t)\phi_\lambda$, so we need to solve the ODE's

$$-\frac{d^2 f_\lambda}{dt^2} + (1+\lambda) f_\lambda = \rho_\lambda,$$

which can be done by standard elementary arguments. The solutions have exponential decay and integration-by-parts is valid, so that

$$\int_{-\infty}^{\infty} \left(\frac{df_\lambda}{dt}\right)^2 + (1+\lambda) f_\lambda^2 \, dt = \int_{-\infty}^{\infty} f_\lambda \rho_\lambda \, dt.$$

Then the Cauchy-Schwartz inequality implies that a

$$\int_{-\infty}^{\infty} f_\lambda^2 \, dt \le \int_{-\infty}^{\infty} \rho_\lambda^2 \, dt,$$

and summing over λ we see that the L^2 norm of the solution f is bounded by that of ρ. Repeated integration by parts shows that for any k there is a constant C_k such that we have $\|f\|_{L^2_{k+2}} \le C_k \|\rho\|_{L^2_k}$ and the statement of the proposition (for $p = 2$) is an easy consequence. (The integration-by-parts argument is made simpler if one uses the fact that on the compact manifold M the L^2_k norm is equivalent to

$$\|g\|_{(k)} = \sum (\lambda+1)^k g_\lambda^2 = \langle g, (\Delta+1)^k g\rangle,$$

for a function $g = \sum g_\lambda \phi_\lambda$.)

With the particular operator $\Delta + 1$ the statement of the Proposition holds for very general class of manifolds, and can be proved in different ways. The advantage of the separation of variables approach above is that it extends easily to other elliptic operators on cylinders.

To tackle nonlinear problems we need Sobolev embedding theorems. These are easy to state.

Proposition 2 *If $k > l, k - n/p > l - n/q$ and $p < q$ then there is a continuous embedding $L^p_k \subset L^q_l$. If $k - n/p > 0$ then there is a continuous embedding $L^p_k \subset C^0$.*

Again, the proofs are not difficult, assuming facts about compact manifolds. Let us just consider the cases which will suffice in our application, when $n = 4$. Then we want to establish embeddings $L^2_1 \subset L^4$ and $L^2_3 \subset C^0$. For the first we use the fact that for functions f on a "band" $M \times [0,1]$ we have an inequality

$$\|f\|_{L^4} \le C \|f\|_{L^2_1}.$$

(This follows from the usual theory for compact manifolds by considering the "double" of the band, i.e. $M \times S^1$.) Now decompose the cylinder $M \times \mathbf{R}$ into a union of copies $\Omega_n = M \times [n, n+1]$ of the band. If f is a function on $M \times \mathbf{R}$ we get

$$\|f\|_{L^4}^4 = \sum \int_{\Omega_n} f^4 \leq C^4 \sum \left(\int_{\Omega_n} |\nabla f|^2 + f^2 \right)^2 \leq C^4 \left(\int_{M \times \mathbf{R}} |\nabla f|^2 + f^2 \right)^2,$$

using the simple fact that for any $a_n \geq 0$ we have

$$\sum a_n^2 \leq \left(\sum a_n \right)^2.$$

The inclusion $L^2_3 \subset C^0$ is even easier–we simply multiply by a standard cut-off function supported in a band. A consequence of these two embeddings is that we have a bounded multiplication map $L^2_3 \times L^2_3 \to L^2_3$.

Now we move on to consider a Riemannian manifold X with cylindrical ends, so the complement of a compact subset of X is isometric to a finite disjoint union of half-cylinders. $M_i \times (0, \infty)$. We consider an operator $\Box$ on X of the form $\Delta_X + V$ where V is a smooth function, equal to 1 on each of the ends. We write $\mathcal{H}_\Box$ for the set of functions f in L^2 with $\Box f = 0$.

Proposition 3 *1. $\mathcal{H}_\Box \subset L^p_k$ for all p, k.*

2. For any p, k the operator $\Box : L^p_{k+2} \to L^p_k$ is Fredholm with kernel $\mathcal{H}_\Box$ and image the orthogonal complement (in the L^2 sense) of $\mathcal{H}_\Box$.

In fact functions in $\mathcal{H}_\Box$ have exponential decay, along with all their derivatives, on the ends of the manifold. Usually one does not encounter manifolds with exactly cylindrical ends but rather ends which are asymptotic to cylinders (as Riemannian manifolds). The extension to this case is completely straightforward.

Now suppose we have a pair X_1, X_2 of such Riemannian manifolds with tubular ends. For simplicity of language, suppose that each has just one end and that the "cross-section" is the same compact manifold M. Given $T > 0$ we form a compact manifold $X_1 \sharp_T X_2$ by gluing the hypersurface corresponding to $M \times \{T\}$ in the end of X_1 to that in the end of X_2, in the obvious way. The result is a Riemannian manifold which contains an isometric copy of $M \times (-T, T)$. Now suppose we have functions V_1, V_2 on X_1, X_2,as above. Then we get a function V and an operator $\Box$ on $X_1 \sharp_T V_2$ in the obvious way. (We use the same symbol $\Box$ to denote the operators on any of the manifolds involved.) The basic fact is

Proposition 4 *Suppose that $\Box$ is invertible on each of X_1, X_2. Then for any p, k there is a constant $C_{p,k}$ and a T_0 such that if $T \geq T_0$ there is a right inverse P to $\Box$ on $X_1 \sharp_T X_2$ and*

$$\|P\rho\|_{L^p_{k+2}} \leq C_{p,k} \|\rho\|_{L^p_k}.$$

The crucial point here is that $C_{p,k}$ does not depend on T, once T is sufficiently large.

The proof of this Proposition is simple. We fix a partition of unity $\gamma_1+\gamma_2 = 1$ on $X_1 \sharp_T X_2$with $\nabla \gamma_i$ supported in a standard band of width 1 in the "middle" of the cylindrical region. Then we choose function β_1, β_2 so that $\beta_i = 1$ on the support of γ_i but β_i is supported in the region which can be considered, by an obvious stretch of langauge, as being contained in X_i. We choose β_i so that $\nabla \beta_i$ is $O(T^{-1})$ and similarly for higher derivatives. Let P_i be the inverse to $\Box$ over X_i and set

$$P_0 \rho = \beta_1 P_1(\gamma_1 \rho) + \beta_2 P_2(\gamma_2 \rho),$$

where again we stretch notation to move functions between X_i and $X_1 \sharp_T X_2$. Then

$$\Box P_0 \rho = \rho + \sum_i 2 \nabla \beta_i \nabla P_i(\gamma_i \rho) + \Delta \beta_i P_i(\gamma_i \rho),$$

and

$$\| \Box P_0 \rho - \rho \|_{L^p_{k+2}} \leq C T^{-1} \| \rho \|_{L^p_k},$$

so when T is large enough we get a genuine right inverse $P_0 \circ (\Box P_0 - 1)^{-1}$ and the estimate of the operator norm of P is immediate.

The Sobolev embedding theorems on the infinite cylinder imply corresponding statements on $X_1 \sharp_T X_2$, with constants independent of T.

2.2 Geometry

We recall some very standard facts about Kahler geometry, the Kummer construction. and the Eguchi-Hanson metric.

Let Z be a complex manifold of complex dimension 2. Giving a Hermitian metric on Z is the same as giving a positive form of type $(1,1)$. The metric is Kahler if this form is closed. Write $\mathcal{D}$ for the operator $2i\overline{\partial}\partial$ mapping (real) functions to (real) forms of type $(1,1)$. If ω is a Kahler form the Laplacian of the metric is given by

$$\Delta_\omega f = (\mathcal{D} f \wedge \omega)/\omega^2,$$

where "division" by the volume form ω^2 has the obvious meaning. Suppose that χ is a nowhere-vanishing holomorphic 2-form on Z. A Kahler metric is Calabi-Yau (i.e. Ricci-flat) if $\omega^2 = \lambda \chi \wedge \overline{\chi}$, for some $\lambda > 0$. If ω_0 is one Kahler form and ϕ is a function then $\omega_\phi = \omega_0 + \mathcal{D}\phi$ is Kahler, provided it is positive (and positivity is an open condition). So we want to solve the Calabi-Yau equation

$$(\omega_0 + \mathcal{D}\phi)^2 = \lambda \chi \wedge \overline{\chi},$$

with the side condition that $\omega_0 + \mathcal{D}\phi > 0$.

Now we turn to the Kummer construction. Let $T^4 = \mathbf{C}^2/\Lambda$ be a complex torus. The map $z \mapsto -z$ on $\mathbf{C}^2$ induces an involution of T^4 with $2^4 = 16$ fixed points. The quotient $\overline{X}$ is an orbifold with 16 singular points, each modelled on the quotient of $\mathbf{C}^2$ by ± 1. We write X for the complement of the singular points in $\overline{X}$. The constant holomorphic 2-form $dz_1 dz_2$ is preserved by by the involution and so descends to a holomorphic form on X.

Consider the map $(z_1, z_2) \mapsto (z_1^2, z_1 z_2, z_2^2) \in \mathbf{C}^3$. This induces a bijection between $\mathbf{C}^2/\pm 1$ and the singular affine quadric in $\mathbf{C}^3$ defined by the equation $v^2 = uw$. We blow-up the origin in $\mathbf{C}^3$ and take the proper transform of this affine surface in the blow-up. The result is a smooth surface Y, the resolution of this singularity. However all we really need to know is that Y is a complex surface which, outside a compact set $K \subset Y$ is identified with the quotient $(\mathbf{C}^2 \setminus B^4)/\pm 1$ and that the holomorphic form on this quotient extends to a nowhere-vanishing form on Y. Making this construction at each of the 16 singular points of X we get a compact complex surface Z, with a nowhere vanishing holomorphic form.

Now our gluing problem will be to find a Calabi-Yau metric on Z starting with standard building blocks: metrics on X and Y. (Of course really we have 16 copies of Y.) The metric, ω_X, on X that we need is just the flat one, but we also need a Calabi-Yau metric on Y. This is the *Eguchi-Hanson* metric, which we will now recall.

Go back to $\mathbf{C}^2$ and write $\rho = r^2 = |z_1|^2 + |z_2|^2$. Consider a Kahler metric of the form $\mathcal{D}\psi$, where $\psi = F(\rho)$. The Calabi-Yau equation becomes

$$\det \begin{pmatrix} F' + |z_1|^2 F'' & F'' z_1 \overline{z}_2 \\ F'' z_2 \overline{z_1} & F' + |z_1|^2 F'' \end{pmatrix} = 1,$$

which is $(F')^2 + \rho F' F'' = 1$. The solution $F(\rho) = \rho$ corresponds to the standard Euclidean metric Ω. Up to re-scalings there is just one other solution which we can take to be given by

$$F'(\rho) = \sqrt{1 + \rho^{-2}}.$$

There is no need to integrate this explicitly, all we need is that, choosing the constant of integration suitably) we have $F(\rho) = \rho + G(\rho)$ where, for $\rho > 1$ $G(\rho)$ has a convergent expansion $a_1 \rho^{-1} + a_2 \rho^{-2} + \ldots$ So we get a Calabi-Yau metric $\Omega + \mathcal{D}G$ on $\mathbf{C}^2 \setminus \{0\}$ where $G = a_1 r^{-2} + a_2 r^{-4} + \ldots$ for $r > 1$. This metric has a singularity at the origin but one can check that when we pass to the quotient and its resolution Y we get a smooth Calabi-Yau metric ω_Y. Choose a positive function r_Y on Y which is equal to $r = \sqrt{|z_1|^2 + |z_2^2}$ on $Y \setminus K$.

To set the scene for the gluing problem, fix a cut-off function β on $\mathbf{R}$, with $\beta(s) = 0$ for $s \leq 1/2$ and $\beta(s) = 1$ for $s \geq 1$. Introduce a (large) parameter R and define a function β_R on Y by $\beta_R = \beta(R^{-1/2} r_Y)$. Put

$$\omega_{R,Y} = \omega_Y - \mathcal{D}(\beta_R G).$$

Then, by construction, $\omega_{R,Y}$ equals the Eguchi-Hanson metric ω_Y when $r_Y \leq \sqrt{R}/2$ and equals the flat metric Ω when $r_Y \geq \sqrt{R}$. The derivatives of G satisfy

$$|\nabla^j G| = O(r_Y^{-2-j}).$$

(Here we measure the size of derivatives with respect to the Euclidean metric.) So on the annulus $\sqrt{R}/2 \leq r_Y \leq \sqrt{R}$ we have $|\nabla^j(G)| = O(R^{-1-j/2})$. The derivatives of β_R satisfy (by scaling)

$$|\nabla^k \beta_R| = O(R^{-k/2}),$$

so any product $\nabla^j \beta_R \nabla^k G$ is $O(R^{-1-(j+k)/2})$. Since $\mathcal{D}(\beta_R G)$ is a sum of such products with $j + k = 2$ we see that

$$|\mathcal{D}(\beta_R G)| = O(R^{-2}).$$

It follows, first, that $\omega_{R,Y}$ is positive (for large enough R) so it is a Kahler metric. Second, we can write

$$\omega_{R,Y}^2 = (1+\eta)^{-1}\omega_Y^2$$

where η is supported on this annulus and $|\eta|$ is $O(R^{-2})$.

Now scale the metric $\omega_{R,Y}$ by a factor R^{-2} (*i.e.* we scale lengths by a factor R^{-1}). The sphere $r_Y = \sqrt{R}$ in Y is then isometric to a small sphere of radius $R^{-1/2}$ about each singular point in X. Take 16 copies of Y, cut out 16 of these balls about the singular points, and glue in the corresponding region in the copies of Y. The result is a Kahler metric ω_0 on the complex manifold Z, which depends on the parameter R. (This parameter can be described more invariantly in terms of the Kahler class.) Our task is to deform this metric—the "approximate solution"—to a genuine Calabi-Yau metric on Z, once the parameter R is sufficiently large.

3 The gluing argument[2]

3.1 Set-up

We want to treat our problem using the "cylindrical ends" theory and to do this we make a conformal change. The basic point is that $\mathbf{C}^2 \setminus \{0\}$ is conformally equivalent to the cylinder $S^3 \times \mathbf{R}$. However the metric on the cylinder is not Kahler. So consider in general a Kahler metric ω and a positive real function h on a complex surface and the conformally equivalent metric $\Theta = h^{-2}\omega$. We write $d\mu$ for the volume form of the metric Θ. Let Q be the differential operator

$$Qf = h\mathcal{D}(h^{-1}f).$$

Notice that Q is not changed if we multiply h by a constant. Set

$$\Box f = (Qf \wedge \Theta)/\Theta^2.$$

Then we have

Lemma 1 *$\Box f = \Delta_\Theta f + Vf$ where $V = h^3 \Delta_\omega(h^{-1})$ and we are writing $\Delta_\Theta, \Delta_\omega$ for the Laplace operators of the two metrics.*

To see this, suppose fmg have compact support and write,

$$\int \Box f g d\mu = \int h\mathcal{D}(h^{-1}f) \wedge g\Theta = \int \mathcal{D}(h^{-1}f)(h^{-1}g)\omega.$$

Now apply Stokes' Theorem and the fact that ω is closed to write this as

$$-2i \int \partial(h^{-1}f)\overline{\partial}(h^{-1}g) \wedge \omega.$$

Some further manipulation, which we leave as an exercise for the reader, shows that this is equal to

$$\int (\nabla f, \nabla g) + V(fg) d\mu,$$

(where the inner product is computed using Θ) with the stated function V.

The conformal equivalence the the flat metric Ω on $\mathbf{C}^2 \setminus \{0\}$ to the cylindrical metric corresponds to $h = r$. We then have

$$\Delta_\Omega r^{-1} = r^{-3} \frac{\partial}{\partial r}(r^3 \frac{\partial r^{-1}}{\partial r}) = r^{-3},$$

so in this case $V = 1$.

Now return to our manifold Z with the Kahler metric ω_0 depending on the parameter R. We have a function r_Y on each copy of Y. Let r_X be a positive function on X which, in a fixed ball about each singular point, is equal to the distance to that singular point. There is then a function h on Z equal to r_X on the "X-side" and to $R^{-1} r_Y$ on the "Y-side". (Since we glued the metrics on the sphere where $r_Y = R^{1/2}, r_X = R^{-1/2}$.) The hermitian metric $\Theta_0 = h^{-1}\omega_0$ contains a long cylinder. More precisely there is a region in Z which we can identify with a cylinder $P^3 \times (-T, T)$ where $P^3 = S^3/\pm 1$ and T is approximately $(\log R)/2$. On this cylinder the co-ordinate $t \in (-T, T)$ is $\log R^{1/2} h$. The metric Θ_0 is precisely cylindrical on the part of the cylinder $t \geq 0$ and is approximately cylindrical on the region $t \leq 0$. We can think of Θ_0 as being obtained in the following way. Define the Hermitian metric $\Theta_X = r_X^{-1}\omega_X$ on X: this has 16 cylindrical ends. Now take the metric $\Theta_Y = r_Y^{-1}\omega_Y$ on Y. This is a metric with an asymptotically cylindrical end. Now "cut-off" the metric Θ_Y at a distance $T/2$ down the end, to make it exactly cylindrical, and perform the "connected sum" construction considered before (except, of course, that we have 16 copies of Y). This "cutting off" is exactly what we have specified above, but we are now viewing it in a slightly different way. We have a differential operator $\Box$ on Z which is equal to $\Delta_{\Theta_0} + V$ where V is equal to 1 in the region $t > 0$ of the cylinder can be supposed to be close to 1 on the region $t < 0$. Again, we have corresponding operators on the complete manifolds X, Y with asymptotically tubular ends. The point of all this is that we can apply our general analytical theory to the operator $\Box$.

3.2 The proof

Everything is now in place to proceed with the proof. We suppose our metric ω_X is chosen so that $\omega_X^2 = \chi \wedge \overline{\chi}$. We seek a function ϕ on Z and $\lambda > 0$ such that

$$(\omega_0 + \mathcal{D}\phi)^2 = \lambda \chi \wedge \overline{\chi}$$

which is to say

$$(\omega_0 + \mathcal{D}\phi)^2 = \lambda(1+\eta)\omega_0^2,$$

Recall that η is defined as $\omega_{R,Y}^2 = (1+\eta)^{-1}\omega_Y^2$. Consider the function η as a function on Z in the obvious way. In the cylindrical picture η is supported on a band $|t| \le \log 2$ say (or, really, 16 such bands, one for each gluing region). We have $|\eta| = O(R^{-2})$ and it is easy to see that the same holds for all derivatives of η. So for any k the L_k^2 norm of η is $O(R^{-2})$. Now write $\phi = hf$ and express the equation in terms of $\Theta_0 = h^{-2}\omega_0$. We get

$$(\Theta_0 + h^{-2}\mathcal{D}(hf))^2 = \lambda(1+\eta)\Theta_0^2.$$

Expanding the quadratic term and rearranging, this is

$$\Box f + h^{-3}Q(f)^2 = h^3(\lambda(1+\eta) - 1).$$

The problem here is that h is very small on the "Y-side", in fact $O(R^{-1})$, so the co-efficient of $Q(f)^2$ is very large. To deal with this, set $f = R^{-3}g$. So we have an equation for the pair (g, λ) which is

$$\Box g + (Rh)^{-3}Q(g)^2 = (Rh)^3(\lambda(1+\eta) - 1).$$

Now $(Rh)^{-1}$ is bounded (along with all its derivatives). The differential operator Q has co-efficients which are bounded, along with all derivatives independent of R. We will solve the equation for g in the Sobolev space L_5^2. Then our multiplication $L_3^2 \times L_3^2 \to L_3^2$ implies that

$$\|(Rh)^{-3}(Q(g_1)^2 - Q(g_2)^2)\|_{L_3^2} \le C\|g_1 - g_2\|_{L_5^2} \left(\|g_1\|_{L_5^2} + \|g_2\|_{L_5^2}\right).$$

Also since $L_3^2 \subset C^0$, a small solution g in L_5^2 will define a positive form. To be precise, small enough so that $C\left(\|g_1\|_{L_5^2} + \|g_2\|_{L_5^2}\right) \le 1$ so we are able to apply contraction argument. The argument works as the following:

Suppose $\Box$ is invertible (which is not true but we'll see it soon that the kernel of $\Box$ on Z is determined) and $C((Rh)^3(\lambda(1+\eta)-1) < \frac{1}{2})$ and consider solution of

$$\Box w + (Rh)^{-3}Q(g)^2 = (Rh)^3(\lambda(1+\eta) - 1).$$

for given g which satisfies $C\|g\|_{L_5^2} < \frac{1}{2}$

So for different solution w_1, w_2 with respect to g_1, g_2, we have

$$\begin{aligned}\|\Box(w_1-w_2)\|_{L^2_3} &= \|(Rh)^{-3}(Q(g_1)^2-Q(g_2)^2)\|_{L^2_3}\\ &\leq C\|g_1-g_2\|_{L^2_5}\left(\|g_1\|_{L^2_5}+\|g_2\|_{L^2_5}\right)\\ &\leq \|g_1-g_2\|_{L^2_5}\end{aligned}$$

By contraction argument, there exists g solves $\Box g+(Rh)^{-3}Q(g)^2=(Rh)^3(\lambda(1+\eta)-1)$. So the problem reduces to solving the linearised equation once we check that the starting point is within the compact set $C\|g\|_{L^2_5}\leq\frac{1}{2}$. That is to check whether we can make

$$(Rh)^3(\lambda(1+\eta)-1)$$

small enough. Since η is supported in a band of fixed width in the middle of the cylinder and on this band Rh is $O(R^{-1/2})$. Since η is $O(R^{-2})$ we see that $(Rh)^3\eta$ is $O(R^{-1/2}) << 1$. The same holds for all derivatives.

It is time to examine the linearised problem which, by our general theory, reduces to considering the kernel of the operator $\Box$ over the complete manifolds X, Y.

By the definition of $\Box$, a function f satisfies $\Box f=0$ if and only if $\Delta_\omega(h^{-1}f)=0$. Consider first f on Y. Then if f tends to zero at infinity the same is true *a fortiori* for $r_Y^{-1}f$ and if $\Delta_\omega(r_Y^{-1}f)=0$ the function must vanish by the maximal principle. So there is no kernel of $\Box$ on Y. Similarly, a function in the kernel of $\Box$ on X corresponds to a harmonic function, in the flat metric, which is $o(r_X^{-1})$ at the singularities. Since the fundamental solution of the Laplacian is 4 dimensions is $O(r^{-2})$ the only possibility is that this function is constant. So there is a 1-dimensional kernel of $\Box$ on X, spanned by the function r_X. Indeed there is obviously a kernel of $\Box$ on Z, spanned by the function h. Thus we are in a slighly more complicated situation than that envisaged before, but the same argument shows that we can invert $\Box$ on Z uniformly "modulo h". That is there is a uniformly bounded operator P and a linear functional π such that

$$\rho=\Box P(\rho)+\pi(\rho)h.$$

This fits in with the fact that we have an additional parameter λ in our problem. Going back to the Kahler picture we know that the metrics ω_0 and $\omega_0+\mathcal{D}\phi$ on Z have the same volume. This goes over to the identity

$$\int h(\Box g+(Rh)^{-3}Q(g)^2)d\mu=0,$$

for any g. Thus the parameter λ is determined by η through the equation

$$\lambda\int_Z(1+\eta)h^4\ d\mu=\int_Z h^4\ d\mu.$$

We define λ by this formula, so $\lambda=1+O(R^{-4})$, since $h=O(R^{-1/2})$ on the support of η. With this value of λ we solve the nonlinear equation "modulo h"

by the inverse function theorem. That is, we solve the equation for (g,τ), where τ is a constant,

$$\Box g + (Rh)^{-3}Q(g)^2 = (Rh)^3(\lambda(1+\eta)-1) + \tau h.$$

Now taking the L^2 inner product with h we see that in fact $\tau = 0$ and we have found our Calabi-Yau metric.

4 Mclean's theorem

4.1 Deformations of compact SL m-folds

The *deformation theory* of special Lagrangian submanifolds was studied by McLean [6, §3], who proved the following result in the Calabi–Yau case.

Theorem 1 *Let N be a compact SL m-fold in an almost Calabi–Yau m-fold (M,J,ω,Ω). Then the moduli space $\mathcal{M}_X$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N.*

We now give a partial proof of Theorem 1 following [4]. We start by recalling some symplectic geometry, which can be found in McDuff and Salamon [7].

Let N be a real m-manifold. Then its tangent bundle T^*N has a *canonical symplectic form* $\hat{\omega}$, defined as follows. Let $(x_1,\ldots,x_m)$ be local coordinates on N. Extend them to local coordinates $(x_1,\ldots,x_m,y_1,\ldots,y_m)$ on T^*N such that $(x_1,\ldots,y_m)$ represents the 1-form $y_1\mathrm{d}x_1+\cdots+y_m\mathrm{d}x_m$ in $T^*_{(x_1,\ldots,x_m)}N$. Then $\hat{\omega} = \mathrm{d}x_1\wedge\mathrm{d}y_1+\cdots+\mathrm{d}x_m\wedge\mathrm{d}y_m$.

Identify N with the zero section in T^*N. Then N is a *Lagrangian submanifold* of T^*N. The *Lagrangian Neighbourhood Theorem* [7, Th. 3.33] shows that any compact Lagrangian submanifold N in a symplectic manifold looks locally like the zero section in T^*N.

Theorem 2 *Let (M,ω) be a symplectic manifold and $N\subset M$ a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood U of the zero section N in T^*N, and an embedding $\Phi: U\to M$ with $\Phi|_N = \mathrm{id}: N\to N$ and $\Phi^*(\omega)=\hat{\omega}$, where $\hat{\omega}$ is the canonical symplectic structure on T^*N.*

In the situation of Theorem 1, let g be the Kähler metric on M, and define $\psi: M\to(0,\infty)$ by

$$\psi^{2m}\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m\Omega\wedge\bar{\Omega}, \tag{1}$$

Applying Theorem 2 gives an open neighbourhood U of N in T^*N and an embedding $\Phi: U\to M$. Let $\pi: U\to N$ be the natural projection. Define an m-form β on U by $\beta = \Phi^*(\mathrm{Im}\,\Omega)$. If α is a 1-form on N let $\Gamma(\alpha)$ be the graph of α in T^*N, and write $C^\infty(U)\subset C^\infty(T^*N)$ for the subset of 1-forms whose graphs lie in U.

Then each submanifold $\tilde{N}$ of M which is C^1-close to N is $\Phi(\Gamma(\alpha))$ for some small $\alpha\in C^\infty(U)$. Here is the condition for $\tilde{N}$ to be special Lagrangian.

Lemma 2 *In the situation above, if $\alpha \in C^\infty(U)$ then $\tilde{N} = \Phi\big(\Gamma(\alpha)\big)$ is a special Lagrangian m-fold in M if and only if* $\mathrm{d}\alpha = 0$ *and* $\pi_*\big(\beta|_{\Gamma(\alpha)}\big) = 0$.

proof, By definition, $\tilde{N}$ is an SL m-fold in M if and only if $\omega|_{\tilde{N}} \equiv \operatorname{Im}\Omega|_{\tilde{N}} \equiv 0$. Pulling back by Φ and pushing forward by $\pi : \Gamma(\alpha) \to N$, we see that $\tilde{N}$ is special Lagrangian if and only if $\pi_*\big(\hat{\omega}|_{\Gamma(\alpha)}\big) \equiv \pi_*\big(\beta|_{\Gamma(\alpha)}\big) \equiv 0$, since $\Phi^*(\omega) = \hat{\omega}$ and $\Phi^*(\operatorname{Im}\Omega) = \beta$. But as $\hat{\omega}$ is the natural symplectic structure on $U \subset T^*N$ we have $\pi_*\big(\hat{\omega}|_{\Gamma(\alpha)}\big) = -\mathrm{d}\alpha$, and the lemma follows.

We rewrite the condition $\pi_*\big(\beta|_{\Gamma(\alpha)}\big) = 0$ in terms of a function F.

Definition 1 *Define $F : C^\infty(U) \to C^\infty(N)$ by $\pi_*\big(\beta|_{\Gamma(\alpha)}\big) = F(\alpha)\,\mathrm{d}V_g$, where $\mathrm{d}V_g$ is the volume form of $g|_N$ on N. Then Lemma 2 shows that if $\alpha \in C^\infty(U)$ then $\Phi\big(\Gamma(\alpha)\big)$ is special Lagrangian if and only if $\mathrm{d}\alpha = F(\alpha) = 0$.*

We compute the expansion of F up to first order in α.

Proposition 5 *This function F may be written*

$$F(\alpha)[x] = -\mathrm{d}^*\big(\psi^m\alpha\big) + Q\big(x, \alpha(x), \nabla\alpha(x)\big) \tag{2}$$

for $x \in N$, where $Q : \big\{(x, y, z) : x \in N,\ y \in T_x^*N \cap U,\ z \in \otimes^2 T_x^*N\big\} \to \mathbb{R}$ is smooth and $Q(x, y, z) = O(|y|^2 + |z|^2)$ for small y, z.

proof. The value of $F(\alpha)$ at $x \in N$ depends on the tangent space $T_{x'}\Gamma(\alpha)$, where $x' \in \Gamma(\alpha)$ with $\pi(x') = x$. But $T_{x'}\Gamma(\alpha)$ depends on both $\alpha|_x$ and $\nabla\alpha|_x$. Hence $F(\alpha)$ depends pointwise on both α and $\nabla\alpha$, rather than just α. So we may take

$$F(\alpha)[x] = -\mathrm{d}^*\big(\psi^m\alpha\big) + Q\big(x, \alpha(x), \nabla\alpha(x)\big)$$

as a definition of Q, and Q is then well-defined on the set of all (x, y, z) realized by $\big(x, \alpha(x), \nabla\alpha(x)\big)$ for $\alpha \in C^\infty(U)$, which is the domain given for Q.

As F depends smoothly on α we see that Q is a smooth function of its arguments. Therefore Taylor's theorem yields

$$Q(x, y, z) = Q(x, 0, 0) + y \cdot (\partial_y Q)(x, 0, 0) + z \cdot (\partial_z Q)(x, 0, 0) + O(|y|^2 + |z|^2)$$

for small y, z. So to prove that $Q(x, y, z) = O(|y|^2 + |z|^2)$ we just need to show that

$$Q(x, 0, 0) = \partial_y Q(x, 0, 0) = \partial_z Q(x, 0, 0) = 0$$

.

Now $N = \Phi(\Gamma(0))$ is special Lagrangian, so $\alpha = 0$ satisfies $F(\alpha) = 0$ by Definition. Thus $F(\alpha)[x] = -\mathrm{d}^*\big(\psi^m\alpha\big) + Q\big(x, \alpha(x), \nabla\alpha(x)\big)$

gives $Q(x, 0, 0) \equiv 0$.

To compute $\partial_y Q(x, 0, 0)$ and $\partial_z Q(x, 0, 0)$, let $\alpha \in C^\infty(U)$ be small, and let v be the vector field on T^*N with $v \cdot \hat{\omega} = -\pi^*(\alpha)$. Then v is tangent to the fibres of $\pi : T^*N \to N$, and $\exp(v)$ maps $T^*N \to T^*N$ taking $\gamma \mapsto \alpha + \gamma$ for 1-forms γ on N. Identifying N with the zero section of T^*N, the image $\exp(sv)[N]$ of N under $\exp(sv)$ is $\Gamma(s\alpha)$ for $s \in [0,1]$.

Therefore $F(s\alpha)\,\mathrm{d}V_g = \exp(sv)^*(\beta)$ for $s \in [0,1]$. Differentiating gives

$$\begin{aligned}
\mathrm{d}F|_0(\alpha)\,\mathrm{d}V_g &= \frac{\mathrm{d}}{\mathrm{d}s}\big(F(s\alpha)\big)\Big|_{s=0}\mathrm{d}V_g \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\big(\exp(sv)^*(\beta)\big)\Big|_{s=0} \\
&= \big(\mathcal{L}_v\beta\big)\Big|_N \\
&= \big(\mathrm{d}(v\cdot\beta) + v\cdot(\mathrm{d}\beta)\big)\Big|_N \\
&= \mathrm{d}\big((v\cdot\beta)|_N\big),
\end{aligned}$$

where $\mathcal{L}_v$ is the Lie derivative, '$\cdot$' contracts together vector fields and forms, and $\mathrm{d}\beta = 0$ as Ω is closed and $\beta = \Phi^*(\mathrm{Im}\,\Omega)$.

Calculation at a point $x \in N$ shows that $(v\cdot\beta)|_N = \psi^m * \alpha$, where $*$ is the Hodge star of g on N. As $*\mathrm{d}V_g = 1$ and $*\mathrm{d}* = -\mathrm{d}^*$ on 1-forms, then we have

$$\mathrm{d}F|_0(\alpha)\,\mathrm{d}V_g = \mathrm{d}(\psi^m * \alpha) = \big(*\mathrm{d}*(\psi^m\alpha)\big)\,\mathrm{d}V_g = \big(-\mathrm{d}^*(\psi^m\alpha)\big)\,\mathrm{d}V_g.$$

Comparing this with $F(\alpha)[x] = -\mathrm{d}^*\big(\psi^m\alpha\big) + Q\big(x, \alpha(x), \nabla\alpha(x)\big)$
shows that $\partial_y Q(x,0,0) = \partial_z Q(x,0,0) = 0$, which completes the proof.

We see that $\mathcal{M}_X$ is locally approximately isomorphic to the vector space of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}^*(\psi^m\alpha) = 0$. But by Hodge theory, this is isomorphic to the de Rham cohomology group $H^1(N,\mathbb{R})$, and is a manifold with dimension $b^1(N)$.

To carry out this last step rigorously requires some technical machinery: one must work with certain *Banach spaces* of sections of $\Lambda^k T^*N$ for $k = 0,1,2$, use *elliptic regularity results* to prove that the map $\alpha \mapsto \big(\mathrm{d}\alpha, \mathrm{d}F|_0(\alpha)\big)$ is *surjective* upon the appropriate Banach spaces, and then use the *Implicit Mapping Theorem for Banach spaces* to show that the kernel of the map is what we expect.

The following rxtended Mclean theorem proved by Marshall [3] is the one we're going to use later.

Theorem 3 *Let $\big\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\big\}$ be a smooth family of deformations of an almost Calabi–Yau m-fold (M, J, ω, Ω), with base space $\mathcal{F} \subset \mathbb{R}^d$. Suppose N is a compact SL m-fold in (M, J, ω, Ω) with $[\omega^s|_N] = 0$ in $H^2(N,\mathbb{R})$ and $[\mathrm{Im}\,\Omega^s|_N] = 0$ in $H^m(N,\mathbb{R})$ for all $s \in \mathcal{F}$. Let $\mathcal{M}_X^{\mathcal{F}}$ be the moduli space of deformations of N in $\mathcal{F}$, and $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ the natural projection.*

Then $\mathcal{M}_X^{\mathcal{F}}$ is a smooth manifold of dimension $d + b^1(N)$, and $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ a smooth submersion. For small $s \in \mathcal{F}$ the moduli space $\mathcal{M}_X^s = (\pi^{\mathcal{F}})^{-1}(s)$ of deformations of N in $(M, J^s, \omega^s, \Omega^s)$ is a nonempty smooth manifold of dimension $b^1(N)$, with $\mathcal{M}_X^0 = \mathcal{M}_X$.

Here a necessary condition for the existence of an SL m-fold $\hat{N}$ isotopic to N in $(M, J^s, \omega^s, \Omega^s)$ is that $[\omega^s|_N] = [\mathrm{Im}\,\Omega^s|_N] = 0$ in $H^*(N,\mathbb{R})$, since $[\omega^s|_N]$

and $[\omega^s|_{\hat{N}}]$ are identified under the natural isomorphism between $H^2(N,\mathbb{R})$ and $H^2(\hat{N},\mathbb{R})$, and similarly for $\mathrm{Im}\,\Omega^s$.

The point of the theorem is that these conditions $[\omega^s|_N] = [\mathrm{Im}\,\Omega^s|_N] = 0$ are also *sufficient* for the existence of $\hat{N}$ when s is close to 0 in $\mathcal{F}$. That is, the only *obstructions* to existence of compact SL m-folds when we deform the underlying almost Calabi–Yau m-fold are the obvious cohomological ones. The version we're going to use is proved by Marshall[3]

Theorem 4 *Let (M, J, g, Ω) be a Calabi–Yau manifold and $(\hat{J}, \hat{g}, \hat{\Omega})$ a deformation of Calabi–Yau struction of (J, g, Ω), with common parameter space the open subset $\mathcal{D} \subseteq \mathbb{R}^m$ containing 0. Suppose that $f : X \to M$ is a compact submanifold which is special Lagrangian with respect to (J, g, Ω) and the $(\hat{J}, \hat{g}, \hat{\Omega})$ satisfies the following:*

$$[f^*\hat{w}(p)] = 0$$

in $H^2(X)$ and

$$[f^* Im\hat{\Omega}(p)] = 0$$

in $H^n(X)$ for each $p \in \mathcal{D}$. Let $N \to X$ be a normal bundle of $f : X \to M$ and identify $N \cong T^*X$ via the bundle isomorphism $\flat_g J$ If $k \geq 2$ then there exists open subsets

$$\begin{aligned} &W \subseteq D \\ &W_1 \subseteq H^1 = \{\xi \in C^{k+1,a}(T^*X) : \delta_g \xi = 0\} \\ &W_2 \subseteq d_g^*(C^{k+2,a}(T^*X)) \oplus d(C^{k+2,a}(T^*X)) \end{aligned}$$

all containing 0 and a smooth map $\chi : W \times W_1 \to W_2$ with $\chi(0) = 0$ such that the following holds:

1. *Every*

$$\begin{aligned} \xi = (\xi_1, \xi_2) &\in W_1 \times W_2 \\ &\subseteq H^1 \oplus d_g^*(C_T^{k+2,a}) \oplus d^*(C^{k+2,a}(T^*X)) \\ &= C^{k+1,a}(T^*X) \\ &\cong C^{k+1,a}(N) \end{aligned}$$

   gives rise to a submanifold $f_\xi : X \to M$ of class $C^{k+1,a}$;

2. *For all $\xi \in (\xi_1, \xi_2) \in W_1 \times W_2$ and $p \in W$ we have*

$$[f_\xi : X \to M \textit{ is special Lagrangian wrt to } (J(p), g(p), \Omega(p))] \iff [\xi_2 = \xi(p, \xi_1)]$$

   and consequently $\chi(W \times W_2) \subseteq d_g^*(C^\infty(T^*X)) \oplus d(C^\infty(T^*X))$

3.

$$M := \{(p,\xi) = (p,\xi_1,\xi_2)$$
$$f_\xi : X \to \; M \textit{ is a special Lagrangian wrt } (J(p), w(p), \Omega(p))\}$$

is a smooth manifold with dimension $dim M_p = b^1(M)s$. *Moreover,*

$$\begin{aligned} W \times W_1 &\to M \\ (p,\xi_1) &\mapsto (p,\xi_1,\chi(p,\xi_1)) \end{aligned}$$

is a diffeomorphism, and the inclusion is a smooth manifold.

4. Given $p \in W$

$$M_p := \{\xi = (\xi_1,\xi_2) \in W_1 \times W_2 : f_\xi : X \to M \textit{ is a special Lagrangian wrt } (J(p), g(p), \Omega(p))\}$$

is a smooth manifold with dimension $dim M_p = b^1(X)$. *Moreover,*

$$\begin{aligned} W_1 &\to M_p \\ \xi_1 &\mapsto (\xi_1,\chi(p,\xi_1)) \end{aligned}$$

is a diffeomorphism and the inclusion $M_p \to M$ *is a smooth manifold.*

5. Given $\xi_1 \in W_1$

$$M_{\xi_1} := \{(p,\xi_2) \in W \times W_2 : f_{\xi_1+\xi_2} : X \to M \textit{ is a special Lagragian wrt } (J(p), g(p), \Omega(p))\}$$

is a smooth manifold with dimension $dim M_{\xi_1} = m$. *Moreover,*

$$\begin{aligned} W &\to M_{\xi_1} \\ p &\mapsto (\xi_1,\chi(p,\xi_1)) \end{aligned}$$

is a diffeomorphism, and the inclusion $M_{\xi_1} \to M$ *is a smooth submanifold.*

5 Existence of Special Lagrangian Sphere

5.1 Special Lagrangian sphere

The singularities on Kummer surface(The Y part mentioned in previous section) is 16 ordinary double point. We're going to show that disingularization of ordinary double point contains a special lagrangian sphere with respect to Eguchi-Hanson metric. For a more detailed discussion on disingularization of ordinary double point, we refer to [5].

Consider cotagent bundle of n-sphere

$$T^*S^n = \{(x,\xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \|x\| = 1, x\cdot \xi = 0\}$$

There exists a diffeomorphism from T^*S^n to the affine quadric

$$Q^n = \{z \in \mathbb{C}^{n+1}, \|z\| = 1\}$$

given by

$$f(x, \xi) = xcosh(|\xi|) + i\frac{sinh(|\xi|)}{|\xi|}\xi$$

Consider the zero section of the affine quadric, i.e. the real sphere. The affine quadric is a calabi-yau manifold with respect to Stenzel metric.[1] which is of the form:

$$\omega = i\partial\bar{\partial}f \circ \tau$$

where τ is the restriction of the function $\|z\|^2 = \Sigma_{j=1}^{n+1}|z_j|^2$ to Q^n. And $f \circ \tau$ is the Ricci-flat Kahler potentials which satisfies the ODE[1]

$$x(f'(x))^n + (f'(x))^{n-1}f''(x)(x^2 - 1) = c > 0$$

The Kahler form $\omega = i\partial\bar{\partial}f \circ \tau = d\alpha$ is exact, where

$$\begin{aligned} \alpha &= -Im\bar{\partial}f \circ \tau \\ &= f'(\tau)\alpha_0 \end{aligned}$$

and $(\alpha_0)_z(v) = \langle Jz, v\rangle$. The holomorphic volume form Ω on Q^n is given by

$$\Omega(v_1, ..., v_n) = 2\Omega_0(z, v_1, ..., v_n)$$

for all $z \in Q^n$, and $v_1, ..., v_n \in T_zQ^n$, where $\Omega_0 = dz_1 \wedge ... \wedge dz_{n+1}$ is the standard holomorphic volume form on $\mathbb{C}^{n+1}$

Since $\alpha|_{S^n} = 0$ so $\omega|_{S^n}$. Also since $Im\Omega$ contains dy^i terms and the tangent space of S^n is spanned by $\frac{\partial}{\partial x^i}$. We conclude that S^n is a special Lagrangian submanifold of T^*S^n

Recall from the end of section 3. The "new" Kahler form is of the form $w_0 + \mathcal{D}\phi$ and since the complex structure remained unchanged so the "new" holomophic volume form is the original holomorphic volume form times a constant. They both satisfy the condition of applying the generalized mclean's theorem:

$$[f^*\hat{w}(p)] = 0, \; [f^*Im\hat{\Omega}(p)] = 0$$

Now we're ready to show that this real two dimensional sphere is the special Lagrangian sphere we're looking for on Kummer surface.

5.2 Conclusion

We know that the real 2–sphere is special Lagrangian with respect to Eguchi–Hanson metric $\omega_Y^2 = (1+\eta)\omega_0^2$ and we can deform it to a Calabi–yau metric $\omega_0 + \mathcal{D}\phi$ on Kummer surface, while the complex structure J remained unchanged.

Recall the calabi-yau matric on Kummer surface is of the form $\omega_0 + \mathcal{D}\phi$, $\phi = hR^{-3}g$ where h is defined as r_X on X and $R^{-1}r_Y$ on Y,

and g is the solution of

$$\Box g + (Rh)^{-3}Q(g)^2 = (Rh)^3(\lambda(1+\eta) - 1).$$

With the convention $C\|g\|_{L^2_5} \leq \frac{1}{2}$, where C is chosen so that

$$\|(Rh)^{-3}(Q(g_1)^2 - Q(g_2)^2)\|_{L^2_3} \leq C\|g_1 - g_2\|_{L^2_5} \left(\|g_1\|_{L^2_5} + \|g_2\|_{L^2_5}\right).$$

So we can choose R large enough to make $|\phi|$ small enough.

By section 3 we know that $\Box f = \Delta_\Theta f + Vf$ where $V = h^3\Delta_\omega(h^{-1})$ is 1 and approximate 1 on each part of Kummer surface. Thus $\|\Box g\|_{L^p_k} \leq C_{p,k}\|g\|_{L^p_{k+2}}$ following the same strategy as proposition 4 in section 2.1. Now expand $\Box$ we get

$$\Box f = h\mathcal{D}(h^{-1}f) \wedge (h^{-2}\omega)/(h^{-2}\omega)^2$$

The leading term (second derivative of f) is of the form $\mathcal{D}f \wedge (h^{-2}\omega)/(h^{-2}\omega)^2$. Compare with the leading term of $\mathcal{D}\phi = \mathcal{D}(hf)$ which is $h\mathcal{D}f$. So we know that $\|\mathcal{D}\phi\|_{L^2_k} \leq C_{p,k}\|\phi\|_{L^2_{k+2}}$ with C doesn't depend on the operator $\mathcal{D}$

Thus by the generalized Mclean's theorem given by Marshall[3] we can see that Kummer surface has special lagrangian sphere.

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